Comparison of Several Tests about the Mean of Normal Distribution in case of known Coefficient of Variation

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Abstract

In this paper, we consider the problem of constructing one sided tests about the mean of the normal population in case of known coefficient of variation. We propose several tests using pivotal method and we compare them numerically via their power functions.

***Keywords*:** Coefficient of variation; normal distribution; pivots; one sided tests; power function.

1. **Introduction**

Normal distribution is widely used to model various phenomena occurring in the fields of agricultural, biological, environmental and physical sciences. In many of such fields, it could be possible to have a statistical population with known coefficient of variation (C.V.). When batches of some substance (chemicals) are to be analyzed and if 'sufficient' batches of the substances are analyzed, their C.V. will be known. In agricultural experiments, it is customary to conduct multi-locational trials. When the results of few centers are available, the C.V. is known and it can be used for the inferential purpose for experiments to be conducted at new locations. In environmental studies, such situations arise when the standard deviation of a pollutant is directly related to the mean.

Several authors have considered the estimation of distribution mean when the population C.V. is known including Marshall (1936), Searls (1964, 1967), Gleaser and Hearly (1976), Sinha (1983), Soofi & Gokhale (1991), Guo & Pal (2003), Wencheko and Wijekoon (2005) and Laheetharan and Wijekoon (2007). Alodat et.al. (2010), constructed several confidence intervals for the mean using the pivotal method in case of normal distribution with known C.V.

Although estimation of the mean has been of interest for many researchers, the question of developing tests for μ in case of known C.V. has received only limited attention. Hinkley (1977) derived two tests for testing the mean of a normal distribution with known C.V. for right alternatives. They are the locally most powerful (LMP) and the conditional tests based on the ancillary statistic for μ. Bhat and Rao (2007) derived the likelihood ratio (LR) and Wald tests for the one- and two-sided alternatives, as well as the two-sided version of the LMP test. The performance of these tests via their power functions are compared with those of the classical t, sign and Wilcoxon signed rank tests. The latter three tests do not use the information on c.v.

It can be noted, as in the above review, that the problem of making statistical inference about a population with known C.V. has significant applications in many fields of science. Moreover, constructing tests about the mean in the case of normal population with known C.V. has not been discussed extensively in the literature. For these reasons, we consider the problem of constructing several tests for the normal population mean in case of known C.V.

Without loss of generality, we may assume that C.V. =1, i.e., we will consider $N(\theta, \theta^2)$ which has the following pdf:

$$f(x \mid \theta) = \frac{1}{\sqrt{2\pi}\theta} e^{-\frac{(x-\theta)^2}{2\theta^2}}, \quad -\infty < x < \infty, \theta > 0$$

In this paper, we will consider testing:

$$H_0 : \theta = \theta_0 \;\; vs \quad H_1 : \theta > \theta_0 \tag{1.1}$$

Without loss of generality we may assume that $\theta_0 = 1$. Then (1.1) will reduce to:

$$H_0 : \theta = 1 \;\; vs \quad H_1 : \theta > 1 \tag{1.2}$$

Note that the uniformly powerful test for (1.2) based on simple random sample does not exist in this case. The following notation will be used in the sequel. We use $X_{(1)},...,X_{(n)}$ to denote the order statistics of a SRS (simple random sample) $X_1,...,X_n$ from $N(\mu,\sigma^2)$, and $Z_{(1)},...,Z_{(n)}$ the order statistics of the corresponding z-scores. The density and the distribution functions of $N(0,1)$ are respectively denoted by $\phi(x)$ and $\Phi(x)$.

2. **The tests**

We will consider the following tests for problem (1.2):

Test 1: Test 1 is based on the pivot $P_1 = \sqrt{n}\left(\frac{(\bar{X}-\theta)}{\theta}\right)$ which has a $N(0,\ 1)$ distribution. The test is given by:

$$\phi_1 = \begin{Bmatrix} 1, \ if \ \sqrt{n}(\bar{X}-1) > Z_\alpha \\ 0\ , \ otherwise \end{Bmatrix} \tag{2.1}$$

where Z_α is the 100(1 - α) quintile of N(0 , 1). The power function of this test is given by:

$$k_1(\theta) = \Phi(\sqrt{n}(\frac{\theta - 1 - \frac{Z_\alpha}{\sqrt{n}}}{\theta}) \tag{2.2}$$

Test 2: Test 2 is based on the pivot $P_2 = \frac{\sqrt{n}(\bar{X}-\theta)}{S}$ which has a t distribution with (n-1) degrees of freedom. This test is given by:

$$\phi_2 = \begin{Bmatrix} 1, \ if \ \frac{\sqrt{n}(\bar{X}-1)}{S} > t_{n-1,\alpha} \\ 0\ , \ otherwise \end{Bmatrix} \tag{2.3}$$

where $t_{n-1,\alpha}$ is the 100(1 - α) quintile of t distribution with (n-1) degrees of freedom. The power function of this test is given by:

$$k_2(\theta) = P_\theta\left(\frac{\sqrt{n}(\bar{X}-1)}{S} > t_{n-1,\alpha}\right) = P_\theta(T > t_{n-1,\alpha}) = 1 - G_2(t_{n-1,\alpha}) \tag{2.4}$$

where $T = \dfrac{\sqrt{n}\left(\frac{\bar{X}-\theta}{\theta}\right) + \sqrt{n}\left(\frac{\theta-1}{\theta}\right)}{\left(\frac{S}{\theta}\right)}$ has a noncentral t distribution with (n-1) degrees of freedom and

noncentrality parameter $\delta = \sqrt{n}\left(\frac{\theta-1}{\theta}\right)$ and G_2 is the cdf of T under θ.

Test 3: Test 3 is based on the pivot $P_3 = \dfrac{(n-1)S^2}{\theta^2}$ which has a chi square distribution with (n-1) degrees of freedom. This test is given by:

$$\phi_3 = \begin{cases} 1, & \text{if } (n-1)S^2 > \chi^2_{n-1,\alpha} \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

where $\chi^2_{n-1,\alpha}$ is the 100(1 - α) quintile of chi square distribution with (n-1) degrees of freedom. The power function of this test is given by:

$$k_3(\theta) = P_\theta((n-1)S^2 > \chi^2_{n-1,\alpha}) = P_\theta\left(\frac{(n-1)S^2}{\theta^2} > \frac{\chi^2_{n-1,\alpha}}{\theta^2}\right) = 1 - G_3\left(\frac{\chi^2_{n-1,\alpha}}{\theta^2}\right) \tag{2.6}$$

where G_3 is the cdf of the chi square distribution with (n-1) degrees of freedom.

Test 4: Test 4 is based on the pivot $P_4 = \dfrac{|X_{(n)}|}{\theta} = |1 + Z_{(n)}|$ where $X_{(n)}$ is the n^{th} order statistic of a SRS of size n from $N(\theta, \theta^2)$) distribution and $Z_{(n)}$ is the n^{th} order statistic of a SRS of size n from N(0,1). This test is given by:

$$\phi_4 = \begin{cases} 1, & \text{if } |X_{(n)}| > \max_\alpha \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

where $\max_\alpha$ is the 100(1 - α) quintile of $|X_{(n)}|$. Note that:

$$\alpha = P_{\theta=1}(|X_{(n)}| > m_\alpha) = 1 - P_{\theta=1}(|X_{(n)}| < m_\alpha)$$

which implies that

$$\begin{aligned} 1-\alpha = P_{\theta=1}(|X_{(n)}| > \max_\alpha) &= P_{\theta=1}(-\max_\alpha < X_{(n)} < \max_\alpha) \\ &= P_{\theta=1}(X_{(n)} < \max_\alpha) - P_{\theta=1}(X_{(n)} < -\max_\alpha) \\ &= \Phi^n(\max_\alpha - 1) - \Phi^n(-\max_\alpha - 1) \end{aligned}$$

The power function of this test is given by:

$$k_4(\theta) = P_\theta(\left|X_{(n)}\right| > \max_\alpha) = P_\theta(\frac{\left|X_{(n)}\right|}{\theta} > \frac{\max_\alpha}{\theta})$$
$$= P(\left|1+Z_{(n)}\right| > \frac{\max_\alpha}{\theta}) = 1 - G_4(\frac{\max_\alpha}{\theta}) \quad (2.8)$$

where G_4 is the cdf of $\left|1+Z_{(n)}\right|$.

Test 5: Test 5 is based on the pivot $P_5 = \frac{|X_{(1)}|}{\theta} = |1+Z_{(1)}|$ where $X_{(1)}$ is the 1st order statistic of a SRS of size n from $N(\theta,\theta^2)$) distribution and $Z_{(1)}$ is the 1st order statistic of a SRS of size n from N(0,1). This test is given by:

$$\phi_5 = \begin{cases} 1, \ if \ |X_{(1)}| > \min_\alpha \\ 0, \ otherwise \end{cases} \quad (2.9)$$

where $\min_\alpha$ is the 100(1 - α) quintile of $|1+Z_{(1)}|$. Note that:

$$\alpha = P_{\theta=1}(\left|X_{(1)}\right| > \min_\alpha) = P_{\theta=1}(X_{(1)} > \min_\alpha) + P_{\theta=1}(X_{(1)} < -\min_\alpha)$$
$$= (1-\Phi(\min_\alpha - 1))^n + 1 - (1-\Phi(-\min_\alpha - 1))^n$$

The power function of this test is given by:

$$k_5(\theta) = P_\theta(\left|X_{(1)}\right| > \min_\alpha) = P_\theta(\frac{\left|X_{(1)}\right|}{\theta} > \frac{\min_\alpha}{\theta})$$
$$= P(\left|1+Z_{(1)}\right| > \frac{\min_\alpha}{\theta}) = 1 - G_5(\frac{\min_\alpha}{\theta}) \quad (2.10)$$

where G_5 is the cdf of $\left|1+Z_{(1)}\right|$.

Test 6: Test 6 is based on the pivot $P_6 = \begin{cases} \frac{|X_{(m)}|}{\theta} = \left|1+Z_{(m)}\right|, & n = 2m-1, \\ \frac{|X_{(m)}+X_{(m+1)}|}{2\theta} = \left|2+Z_{(m)}+Z_{(m+1)}\right| & n = 2m. \end{cases}$

where $X_{(i)}$ is the i^{th} order statistic of a SRS of size n from $N(\theta,\theta^2)$) distribution and $Z_{(i)}$ is the i^{th} order statistic of a SRS of size n from N(0,1). This test is given by:

$$\phi_6 = \begin{cases} 1, \ if \ \left|X_{(m)}\right| > \mathrm{mi}d_\alpha \ and \ n = 2m-1 \\ 1, \ if \ \left|\frac{X_{(m)}+X_{(m+1)}}{2}\right| > \mathrm{mi}d_\alpha \ and \ n = 2m \\ 0, \ otherwise \end{cases} \quad (2.11)$$

where $\mathrm{mi}d_\alpha$ is the 100(1 - α) quintile of P_6 under $\theta = 1$. The power function of this test is given by:

$$k_6(\theta) = P_\theta(P_6 > \text{mi}d_\alpha) = \begin{cases} P_\theta(\dfrac{|X_{(m)}|}{\theta} = \dfrac{|1+Z_{(m)}|}{\theta} > \dfrac{\text{mi}d_\alpha}{\theta}) \; if \;\; n = 2m-1 \\ P_\theta(\dfrac{\left|\frac{X_{(m)}+X_{(m+1)}}{2}\right|}{\theta} = \dfrac{\left|\frac{Z_{(m)}+Z_{(m+1)}+2}{2}\right|}{\theta} > \dfrac{\text{mi}d_\alpha}{\theta}) \; if \;\; n = 2m \end{cases} \tag{2.12}$$

$$= \begin{cases} 1 - G_{6,odd}(\dfrac{\text{mi}d_\alpha}{\theta}) \;\; if \;\; n = 2m-1 \\ 1 - G_{6,even}(\dfrac{\text{mi}d_\alpha}{\theta}) \;\; if \;\; n = 2m \end{cases}$$

where $G_{6,odd}$ is the cdf of $|1+Z_{(m)}|$ if n=2m-1 and $G_{6,even}$ is the cdf of $\left|\dfrac{Z_{(m)}+Z_{(m+1)}+2}{2}\right|$ if n=2m

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Test 7: Test 7 is based on the pivot $P_7 = \dfrac{X_{(n)} - X_{(1)}}{\theta} = Z_{(n)} - Z_{(1)}$. This test is given by:

$$\phi_7 = \begin{cases} 1, \; if \; X_{(n)} - X_{(1)} > c_\alpha \\ 0 \,, \; otherwise \end{cases} \tag{2.13}$$

where c_α is the 100(1 - α) quintile of $Z_{(n)} - Z_{(1)}$. The power function of this test is given by:

$$\begin{aligned} k_7(\theta) &= P_\theta(X_{(n)} - X_{(1)} > c_\alpha) = P_\theta(\frac{X_{(n)} - X_{(1)}}{\theta} > \frac{c_\alpha}{\theta}) \\ &= P(Z_{(n)} - Z_{(1)} > \frac{c_\alpha}{\theta}) = 1 - G_7(\frac{c_\alpha}{\theta}) \end{aligned} \tag{2.14}$$

where G_7 is the cdf of $Z_{(n)} - Z_{(1)}$.

Test 8: Test 8 is based on the pivot $P_8 = \left|\dfrac{X_{(n)} + X_{(1)}}{\theta}\right| = |Z_{(n)} + Z_{(1)} + 2|$. This test is given by:

$$\phi_8 = \begin{cases} 1, \; if \; |X_{(n)} + X_{(1)}| > d_\alpha \\ 0 \,, \; otherwise \end{cases} \tag{2.15}$$

where d_α is the 100(1 - α) quintile of $|Z_{(n)} + Z_{(1)} + 2|$. The power function of this test is given by:

$$\begin{aligned} k_8(\theta) &= P_\theta(|X_{(n)} + X_{(1)}| > d_\alpha) = P_\theta(\frac{|X_{(n)} + X_{(1)}|}{\theta} > \frac{d_\alpha}{\theta}) \\ &= P(|Z_{(n)} + Z_{(1)} + 2| > \frac{d_\alpha}{\theta}) = 1 - G_8(\frac{d_\alpha}{\theta}) \end{aligned} \tag{2.16}$$

where G_8 is the cdf of $\left|Z_{(n)} + Z_{(1)} + 2\right|$.

Test 9: Test 9 is based on the pivot $P_9 = \frac{X_{(n)} - X_{(m)}}{\theta} = Z_{(n)} - Z_{(m)}$. This test is given by:

$$\phi_9 = \begin{Bmatrix} 1, \ if \ \ X_{(n)} - X_{(m)} > b_\alpha \\ 0 \ , \ otherwise \end{Bmatrix} \tag{2.17}$$

where b_α is the 100(1 - α) quintile of $Z_{(n)} - Z_{(m)}$. The power function of this test is given by:

$$\begin{aligned} k_9(\theta) &= P_\theta(X_{(n)} - X_{(m)} > e_\alpha) = P_\theta(\frac{X_{(n)} - X_{(m)}}{\theta} > \frac{e_\alpha}{\theta}) \\ &= P(Z_{(n)} - Z_{(m)} > \frac{e_\alpha}{\theta}) = 1 - G_9(\frac{e_\alpha}{\theta}) \end{aligned} \tag{2.18}$$

where G_9 is the cdf of $Z_{(n)} - Z_{(m)}$.

Test 10: Test 10 is based on the pivot $P_{10} = \left|\frac{X_{(1)}X_{(n)}}{\theta^2}\right| = \left|(1+Z_{(1)})(1+Z_{(n)})\right|$. This test is given by:

$$\phi_{10} = \begin{Bmatrix} 1, \ if \ \ \left|X_{(1)}X_{(n)}\right| > f_\alpha \\ 0 \ , \ otherwise \end{Bmatrix} \tag{2.19}$$

where f_α is the 100(1 - α) quintile of $\left|(1+Z_{(1)})(1+Z_{(n)})\right|$. The power function of this test is given by:

$$\begin{aligned} k_{10}(\theta) &= P_\theta(\left|X_{(1)}X_{(n)}\right| > f_\alpha) = P_\theta(\frac{\left|X_{(1)}X_{(n)}\right|}{\theta^2} > \frac{f_\alpha}{\theta^2}) \\ &= P(\left|Z_{(1)}Z_{(n)}\right| > \frac{f_\alpha}{\theta^2}) = 1 - G_{10}(\frac{f_\alpha}{\theta^2}) \end{aligned} \tag{2.20}$$

where G_{10} is the cdf of $\left|Z_{(1)}Z_{(n)}\right|$.

Test 11: Test 11 is based on the pivot $P_{11} = \left|\frac{X_1 \ldots X_n}{\theta^n}\right| = \left|(1+Z_1)\ldots.(1+Z_n)\right|$. This test is given by:

$$\phi_{11} = \begin{Bmatrix} 1, \ if \ \ \left|X_1 \ldots X_n\right| > g_\alpha \\ 0 \ , \ otherwise \end{Bmatrix} \tag{2.21}$$

where g_α is the 100(1 - α) quintile of $\left|(1+Z_1)\ldots.(1+Z_n)\right|$. The power function of this test is given by:

$$\begin{aligned} k_{11}(\theta) &= P_\theta(|X_1...X_n| > g_\alpha) = P_\theta(\frac{|X_1...X_n|}{\theta^n} > \frac{g_\alpha}{\theta^n}) \\ &= P(|Z_1...Z_n| > \frac{g_\alpha}{\theta^n}) = 1 - G_{11}(\frac{g_\alpha}{\theta^n}) \end{aligned} \quad (2.22)$$

where G_{11} is the cdf of $|(1+Z_1)....(1+Z_n)|$.

Test 12: Test 12 is based on the pivot

$P_{12} = \frac{|X_1|^k + |X_2|^k + ...|X_n|^k}{\theta^k} = |1+Z_1|^k + |1+Z_2|^k + ...|1+Z_n|^k$ where k>0. This test is given by:

$$\phi_{12} = \begin{cases} 1, \ if \ |X_1|^k + |X_2|^k + ...|X_n|^k > k_\alpha \\ 0 \ , \ otherwise \end{cases} \quad (2.23)$$

where k_α is the 100(1 - α) quintile of $|1+Z_1|^k + |1+Z_2|^k + ...|1+Z_n|^k$. The power function of this test is given by:

$$\begin{aligned} k_{12}(\theta) &= P_\theta(|X_1|^k + |X_2|^k + ...|X_n|^k > k_\alpha) = P_\theta(\frac{|X_1|^k + |X_2|^k + ...|X_n|^k}{\theta^k} > \frac{k_\alpha}{\theta^k}) \\ &= P(|1+Z_1|^k + |1+Z_2|^k + ...|1+Z_n|^k > \frac{k_\alpha}{\theta^k}) = 1 - G_{12}(\frac{k_\alpha}{\theta^k}) \end{aligned} \quad (2.24)$$

where G_{12} is the cdf of $|1+Z_1|^k + |1+Z_2|^k + ...|1+Z_n|^k$.

Test 13: Test 13 is based on the pivot $P_{13} = -2\sum_{i=1}^{n} In(\Phi\left(\frac{X_i - \theta}{\theta}\right)) = -2\sum_{i=1}^{n} In(\Phi(Z_i))$. This test is called the Fisher nonparametric test and is given by:

$$\phi_{13} = \begin{cases} 1, \ if \ \ -2\sum_{i=1}^{n} In(1-\Phi(X_i - 1)) > \chi^2_{2n,\alpha} \\ 0 \ , \ otherwise \end{cases} \quad (2.25)$$

where $\chi^2_{2n,\alpha}$ is the 100(1 - α) quintile of chi square distribution with 2n degrees of freedom. The power function of this test is given by:

$$\begin{aligned} k_{13}(\theta) &= P_\theta(-2\sum_{i=1}^{n} In(1-\Phi(X_i - 1)) > \chi^2_{2n,\alpha}) \\ &= 1 - G_{13}(\chi^2_{2n,\alpha}) \end{aligned} \quad (2.26)$$

where G_{13} is the cdf of $-2\sum_{i=1}^{n} In(1-\Phi(X_i - 1))$ under θ .

Test 14: Test 14 is based on the pivot $P_{14} = -\sum_{i=1}^{n} In(\frac{1-\Phi\left(\frac{X_i-\theta}{\theta}\right)}{\Phi\left(\frac{X_i-\theta}{\theta}\right)}) = -\sum_{i=1}^{n} In(\frac{1-\Phi(Z_i)}{\Phi(Z_i)})$.

This test is called the logistic nonparametric test and is given by:

$$\phi_{14} = \begin{cases} 1, \ if \ -\sum_{i=1}^{n} In(\frac{1-\Phi(X_i-1)}{\Phi(X_i-1)}) > l_\alpha \\ 0 \ , \ otherwise \end{cases} \tag{2.27}$$

where l_α is the 100(1 - α) quintile of W is a random variable which results from convolving n independent standard logistic distributions. The power function of this test is given by:

$$\begin{aligned} k_{14}(\theta) &= P_\theta(-\sum_{i=1}^{n} In(\frac{1-\Phi(X_i-1)}{\Phi(X_i-1)}) > l_\alpha) \\ &= 1 - G_{14}(l_\alpha) \end{aligned} \tag{2.28}$$

where G_{14} is the cdf of $-\sum_{i=1}^{n} In(\frac{1-\Phi(X_i-1)}{\Phi(X_i-1)})$ under θ.

Test 15: Test 15 is based on the pivot $P_{15} = -\sum_{i=1}^{n}(1-\Phi\left(\frac{X_i-\theta}{\theta}\right)) = -\sum_{i=1}^{n}(1-\Phi(Z_i))$. This test is called the sum of the p-values nonparametric test and is given by:

$$\phi_{15} = \begin{cases} 1, \ if \ -\sum_{i=1}^{n}(1-\Phi(X_i-1)) > s_\alpha \\ 0 \ , \ otherwise \end{cases} \tag{2.29}$$

where s_α is the 100(1 - α) quintile of W is a random variable which results from convolving n independent uniformly distributions over the interval (0,1). The power function of this test is given by:

$$\begin{aligned} k_{15}(\theta) &= P_\theta(-\sum_{i=1}^{n}(1-\Phi(X_i-1)) > s_\alpha) \\ &= 1 - G_{15}(s_\alpha) \end{aligned} \tag{2.30}$$

where G_{15} is the cdf of $-\sum_{i=1}^{n}(1-\Phi(X_i-1))$ under θ.

Test 16: Test 16 is based on the pivot

$P_{16} = \min(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, ..., \frac{X_n-\theta}{\theta}) =^d m\ in(Z_1, Z_2, ..., Z_n)$. This test is given by:

$$\phi_{16} = \begin{cases} 1, \ \mathit{if} \ \ mi\ n(X_1, X_2, ..., X_n) > \min_\alpha \\ 0\ , \ otherwise \end{cases} \tag{2.31}$$

where $\min_\alpha$ is the 100(1 - α) quintile of $m\ in(Z_1, Z_2, ..., Z_n)$. Note that:

$$\alpha = P_{\theta=1}(m\ in(X_1, X_2, ..., X_n) > t_\alpha) = P_{\theta=1}(\min(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, ..., \frac{X_n-\theta}{\theta} > \frac{\min_\alpha - \theta}{\theta})$$

$$= P(m\ in(Z_1, Z_2, ..., Z_n) > \min_\alpha - 1) = (1 - \Phi(\min_\alpha - 1))^n$$

which implies that

$$\min_\alpha = 1 + \Phi^{-1}(1 - \alpha^{1/n})$$

The power function of this test is given by:

$$\begin{aligned} k_{16}(\theta) &= P_\theta(\min(X_1, X_2, ..., X_n) > 1 + \Phi^{-1}(1-\alpha^{1/n})) \\ &= P_\theta((\min(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, ..., \frac{X_n-\theta}{\theta} > \frac{1+\Phi^{-1}(1-\alpha^{1/n}-\theta}{\theta}) \\ &= P((\min(Z_1, Z_2, ..., Z_n > \frac{1+\Phi^{-1}(1-\alpha^{1/n}-\theta}{\theta}) \\ &= (1-\Phi(\frac{1+\Phi^{-1}(1-\alpha^{1/n}-\theta}{\theta}))^n \end{aligned} \tag{2.32}$$

Test 17: Test 17 is based on the pivot

$P_{17} = \max(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, ..., \frac{X_n-\theta}{\theta}) =^d \max(Z_1, Z_2, ..., Z_n)$. This test is given by:

$$\phi_{17} = \begin{cases} 1, \ \mathit{if} \ \ \max(X_1, X_2, ..., X_n) > \max_\alpha \\ 0\ , \ otherwise \end{cases} \tag{2.33}$$

where $\max_\alpha$ is the 100(1 - α) quintile of $\max(Z_1, Z_2, ..., Z_n)$. Note that:

$$\alpha = P_{\theta=1}(\max(X_1, X_2, ..., X_n) > \max_\alpha) = P_{\theta=1}(\max(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, ..., \frac{X_n-\theta}{\theta} > \frac{\max_\alpha - \theta}{\theta})$$

$$= P(\max(Z_1, Z_2, ..., Z_n) > \max_\alpha - 1) = 1 - (\Phi(\max_\alpha - 1))^n$$

which implies that

$$\max_\alpha = 1 + \Phi^{-1}(1-\alpha)^{1/n}$$

The power function of this test is given by:

$$\begin{aligned} k_{17}(\theta) &= P_\theta(\max(X_1, X_2, \ldots, X_n) > 1 + \Phi^{-1}((1-\alpha)^{1/n})) \\ &= P((\max(\frac{X_1-\theta}{\theta}, \frac{X_2-\theta}{\theta}, \ldots, \frac{X_n-\theta}{\theta} > \frac{1+\Phi^{-1}((1-\alpha^{1/n}))-\theta}{\theta}) \\ &= P((\max(Z_1, Z_2, \ldots, Z_n > \frac{1+\Phi^{-1}((1-\alpha^{1/n}))-\theta}{\theta}) \\ &= 1-\Phi((\frac{1+\Phi^{-1}((1-\alpha^{1/n}))-\theta}{\theta}))^n \end{aligned} \tag{2.34}$$

Test 18: Test 18 is the locally most powerful test which was derived by Hinkley (1977) and h is given by:

$$\begin{aligned} \phi_{18} &= \begin{cases} 1, & if \ \frac{\partial}{\partial\theta} In(L(\theta, x_1 \ldots x_n)|_{\theta=1} > c \\ 0, & otherwise \end{cases} \\ &= \begin{cases} 1, & if \ \sum_{i=1}^{n}(x_i - \frac{1}{2})^2 > q_\alpha \\ 0, & otherwise \end{cases} \end{aligned} \tag{2.35}$$

where q_α is the 100(1 - α) quintile of $\sum_{i=1}^{n}(x_i - \frac{1}{2})^2$ under $\theta = 1$. The distribution of $\sum_{i=1}^{n}(x_i - \frac{1}{2})^2$ under $\theta = 1$ is non-central chi-square distribution with n degrees of freedom and non-centrality parameter n/4. The power function of this test is given by:

$$k_{18}(\theta) = P_\theta(\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i > q_\alpha) = 1 - G_{18}(q_\alpha) \tag{2.36}$$

where G_{18} is the cdf of $\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i$ under θ.

Test 19: The likelihood function is given by:

$$L(\theta, x_1 \ldots x_n) = \prod_{i=1}^{n}(\frac{e^{-\frac{(x_i-\theta)^2}{2\theta^2}}}{\theta\sqrt{2\pi}}) = \frac{e^{-\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2\theta^2}}}{\theta^n (2\pi)^{n/2}}$$

which implies that the likelihood function is given by:

$$InL(\theta, x_1 \ldots x_n) = \frac{-\sum_{i=1}^{n} x_i^2}{2\theta^2} + \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n}{2} - nIn\theta - (n/2)In(2\pi)$$

which in turn implies that

$$\frac{\partial}{\partial\theta}(InL(\theta, x_1 \ldots x_n)) = \frac{\sum_{i=1}^{n} x_i^2}{\theta^3} - \frac{\sum_{i=1}^{n} x_i}{\theta^2} - \frac{n}{\theta}$$

Test 19 is the likelihood ratio test which is given by:

$$\phi_{19} = \begin{cases} 1, \ if \ \dfrac{\sup_{\theta\geq 1}(In(L(\theta, x_1....x_n))}{In(L(\theta=1, x_1....x_n)} > d \\ 0\ , \ otherwise \end{cases}$$

$$= \begin{cases} 1, \ if \ \dfrac{(In(L(\theta=\widehat{\theta}, x_1....x_n))}{In(L(\theta=1, x_1....x_n)} > d \\ 0\ , \ otherwise \end{cases} \quad (2.37)$$

$$= \begin{cases} 1, \ if \ \sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i > q_\alpha \\ 0\ , \ otherwise \end{cases}$$

where $\widehat{\theta}$ is the MLE of $\theta \geq 1$. The likelihood function where q_α is the 100(1 - α) quintile of $\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i$ under $\theta = 1$. The power function of this test is given by:

$$k_{19}(\theta) = P_\theta(\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i > q_\alpha\ ) = 1 - G_{17}(q_\alpha) \quad (2.38)$$

where G_{19} is the cdf of $\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i$ under θ.

Numerical comparisons

To compare the 19 tests that are proposed in the previous section, we conducted a simulation study to obtain the critical point (for those whose critical points are not given above) and the power of each of these tests where the significance level of these tests is set at $\alpha = 0.05$. The outline of simulation for a test which rejects H_0 for if T is greater than c as follows:

(1) Simulated 10000 sample from N(1,1) and we computed the value of T for each of these samples, say $T_1, T_2, ..., T_{10000}$.

(2) Arrange $T_1, T_2, ..., T_{10000}$ in an increasing order as $T_{(1)}, T_{(2)}, ..., T_{(10000)}$. Then $c = T_{(9500)}$.

(3) Simulated 5000 sample from $N(\theta, \theta^2)$) and we computed the value of T for each of these samples, say $T_1, T_2, ..., T_{10000}$. Then the percentage of $T_1, T_2, ..., T_{10000}$ who are larger than c will the power of the test at θ, where $\theta > 1$.

The power of the tests were calculated for $\theta = 1.2, 1.4, 1.6, 1.8, 2, 2.2$ and n=5, 10, 15, 20, 25 and 30. The results are reported in Tables 1a-c below. For the test 12, we considered k=0.5, 1, 2, 3 and 6. In the tables, the power of the tests for these values of k is denoted by Test12a, Test12b, Test12c, Test12d and Test12e respectively.

Table 1a: Simulated power of the proposed tests.

	θ=1.2	θ=1.4

	5	10	15	20	25	30	5	10	15	20	25	30
test01	.1591	.1994	.2342	.2659	.2955	.3235	.2960	.3930	.4728	.5410	.6001	.6517
test02	.7998	.8597	.8871	.8727	.8599	.8239	.8261	.8704	.8222	.9365	.9408	.9136
test03	.6784	.6864	.7659	.7714	.8018	.7975	.7554	.8567	.8981	.8815	.9252	.9299
test04	.7049	.7225	.7148	.8036	.7625	.7265	.7850	.8613	.9081	.9201	.9416	.9194
test05	.5216	.5620	.5602	.6027	.6042	.6070	.5848	.5637	.6331	.6098	.6664	.6883
test06	.5921	.5978	.6964	.7062	.7115	.7298	.6594	.7334	.7853	.8228	.8220	.8533
test07	.6209	.7263	.7430	.7344	.7469	.7240	.7167	.7911	.8539	.9016	.9089	.9219
test08	.5852	.5594	.6230	.6511	.6383	.6461	.7025	.7122	.7619	.7407	.7420	.7550
test09	.6213	.5864	.6092	.6639	.6725	.6978	.6528	.7142	.7564	.7926	.7819	.8054
test10	.5833	.5908	.6236	.6278	.6523	.7324	.7193	.7167	.7435	.7578	.7758	.8107
test11	.6369	.6667	.7137	.7315	.7620	.7707	.7155	.7671	.8058	.8540	.8653	.9122
test12a	.6908	.7075	.7715	.7962	.8006	.8483	.7337	.8433	.8925	.9224	.9309	.9330
test12b	.7103	.6845	.7897	.7971	.8499	.8231	.8026	.8643	.8921	.9299	.9494	.9742
test12c	.7097	.7157	.7944	.8157	.8720	.8655	.7898	.8874	.9193	.9552	.9715	.9817
test12d	.6601	.7373	.7834	.8239	.8653	.8496	.7698	.9071	.9365	.9520	.9663	.9812
test12e	.6702	.7080	.7393	.8032	.8503	.8544	.7874	.8911	.9157	.9476	.9498	.9651
test13	.6328	.7331	.7678	.7787	.8454	.8602	.7908	.8956	.9086	.9502	.9613	.9698
test14	.6388	.6540	.7191	.7107	.7356	.7452	.7058	.7898	.8036	.8322	.8833	.9059
test15	.6408	.6955	.7342	.7638	.7896	.8117	.7310	.8081	.8554	.8895	.9142	.9329
test16	.4979	.4538	.4367	.4153	.4013	.3856	.5021	.4295	.4004	.3678	.3372	.2858
test17	.7409	.7025	.7211	.7689	.8051	.7757	.7764	.8374	.9067	.8858	.9290	.9232
test18	.6769	.7544	.8052	.8043	.8659	.8761	.8200	.8729	.9340	.9645	.9758	.9848
test19	.6419	.7295	.7552	.7886	.8429	.8486	.7952	.8715	.9201	.9409	.9529	.9752

Table 1b: Simulated power of the proposed tests.

	θ=1.6						θ=1.8					
	5	10	15	20	25	30	5	10	15	20	25	30
test01	.4248	.5627	.6643	.7418	.8015	.8475	.5319	.6885	.7903	.8585	.9046	.9358
test02	.8708	.9295	.8946	.9137	.9092	.9282	.9243	.9370	.9369	.9533	.9392	.9310
test03	.8792	.9246	.9589	.9656	.9881	.9846	.8813	.9620	.9741	.9923	.9933	.9979
test04	.8261	.9293	.9484	.9704	.9855	.9838	.9069	.9671	.9807	.9910	.9925	.9955
test05	.6605	.6404	.6412	.7340	.7147	.6969	.6407	.6648	.7250	.7471	.7703	.7819
test06	.7333	.7942	.8609	.8912	.9223	.9142	.7645	.8675	.8834	.9242	.9500	.9486
test07	.8089	.9074	.9466	.9458	.9550	.9721	.8751	.9335	.9697	.9792	.9828	.9957
test08	.7305	.7932	.8151	.8248	.8135	.8335	.8042	.8119	.8422	.8424	.8524	.8347
test09	.7103	.7717	.8239	.8301	.8962	.9224	.7534	.8501	.8843	.9275	.9171	.9454
test10	.7399	.7798	.7587	.8061	.8627	.8913	.8149	.7884	.8504	.8888	.8946	.9122
test11	.7818	.8560	.8889	.9134	.9226	.9448	.8141	.9001	.9299	.9649	.9768	.9749
test12a	.8023	.9055	.9542	.9620	.9800	.9850	.8756	.9430	.9788	.9880	.9966	.9986
test12b	.8449	.9231	.9643	.9891	.9918	.9956	.9008	.9650	.9902	.9964	.9982	.9995
test12c	.8633	.9540	.9776	.9928	.9949	.9983	.9188	.9682	.9938	.9979	.9995	.9997

test12d	.8877	.9400	.9805	.9926	.9970	.9982	.9141	.9777	.9947	.9981	.9997	.9999
test12e	.8798	.9355	.9733	.9854	.9914	.9899	.9177	.9707	.9920	.9961	.9987	.9991
test13	.8462	.9308	.9675	.9812	.9933	.9946	.8952	.9669	.9910	.9974	.9983	.9996
test14	.7691	.8329	.8864	.9310	.9523	.9683	.8163	.8930	.9370	.9613	.9764	.9854
test15	.7875	.8697	.9158	.9443	.9621	.9741	.8254	.9069	.9470	.9690	.9815	.9888
test16	.4699	.3710	.3248	.2773	.2937	.2140	.4727	.3569	.3288	.3015	.2561	.2160
test17	.8260	.9163	.9671	.9651	.9740	.9781	.8949	.9603	.9846	.9867	.9927	.9974
test18	.8891	.9603	.9849	.9925	.9969	.9974	.9301	.9837	.9936	.9989	.9998	1.000
test19	.8430	.9322	.9750	.9862	.9932	.9954	.8960	.9572	.9843	.9965	.9981	.9993

Table 1c: Simulated power of the proposed tests.

	θ=2.0						θ=2.2					
	5	10	15	20	25	30	5	10	15	20	25	30
test01	.6162	.7760	.8674	.9213	.9533	.9723	.6815	.8358	.9139	.9546	.9761	.9875
test02	.9107	.9521	.9404	.9595	.9417	.9525	.8938	.9471	.9711	.9658	.9674	.9864
test03	.9355	.9814	.9933	.9968	.9987	.9974	.9474	.9928	.9968	.9983	.9997	.9992
test04	.9287	.9802	.9952	.9963	.9973	.9986	.9515	.9895	.9962	.9976	.9998	.9996
test05	.6851	.6669	.7551	.7660	.8178	.8487	.7083	.7114	.7594	.8057	.8439	.8653
test06	.8237	.8817	.9221	.9442	.9577	.9817	.8310	.9158	.9445	.9665	.9802	.9847
test07	.9109	.9562	.9855	.9929	.9958	.9986	.9192	.9814	.9947	.9958	.9990	.9994
test08	.8140	.8803	.8682	.8793	.8899	.8902	.8393	.8700	.8893	.8880	.9212	.9276
test09	.7847	.8824	.9159	.9364	.9634	.9756	.8220	.9165	.9453	.9662	.9892	.9893
test10	.8254	.8400	.8752	.9035	.9379	.9580	.8405	.8911	.8894	.9213	.9407	.9622
test11	.8815	.9261	.9617	.9778	.9864	.9965	.8735	.9533	.9742	.9874	.9965	.9984
test12a	.9225	.9694	.9897	.9965	.9978	.9995	.9411	.9832	.9973	.9985	.9995	.9999
test12b	.9335	.9825	.9955	.9987	.9997	.9999	.9419	.9914	.9980	.9996	.9999	1.000
test12c	.9523	.9879	.9990	.9992	.9999	1.000	.9693	.9960	.9993	.9999	1.000	1.000
test12d	.9555	.9915	.9982	.9997	1.000	1.000	.9686	.9950	.9983	.9999	1.000	1.000
test12e	.9430	.9906	.9966	.9993	.9997	1.000	.9611	.9944	.9996	.9997	1.000	1.000
test13	.9304	.9830	.9965	.9985	.9997	.9999	.9459	.9924	.9979	.9997	.9999	1.000
test14	.8408	.9256	.9621	.9759	.9899	.9934	.8674	.9447	.9790	.9893	.9939	.9978
test15	.8519	.9302	.9640	.9814	.9900	.9945	.8709	.9446	.9742	.9879	.9940	.9970
test16	.4578	.3512	.2732	.2294	.1923	.1709	.4629	.3341	.2425	.1787	.1599	.1420
test17	.9317	.9847	.9905	.9963	.9963	.9983	.9395	.9919	.9979	.9978	.9992	.9996
test18	.9535	.9937	.9984	.9999	1.000	1.000	.9693	.9974	.9997	1.000	1.000	1.000
test19	.9315	.9815	.9964	.9983	.9997	.9999	.9464	.9919	.9985	.9998	1.000	1.000

5. Numerical comparison

From the tables above, we notice that the power of each of the tests increases as n increases for fixed value of θ and increases as θ increases for fixed value of n except for Test16. As expected, the power is smal for low n and theta close to the null hypothesis' theta, which is one in this study. Test12 consistently gave high power for all values of k. The best is when k is equal to 6. The next best tests are Test18 and Test13. The worst tests are Test16, Test05, Test01 and Test08.Finally, in spite of limitation to the normal distribution, it can be easily extended to the location family with known coefficient of variation.

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